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λ_c-Separation Axioms Via λ_c-open sets

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Abstract. In this paper, we define new types of separation axioms which we call λ_c-T_i ($i = 0, ½, 1, 2$), and characterize these spaces by using the notion of λ_c-closed and λ_c-open sets.

Keywors. λ_c-open **set,** λ_c-closed set**,** λ_c-T_i spaces.

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1. Introduction

Ahmad and Hussain [1] continued to study the properties of γ-operations on topological spaces introduced by Kasahara[2]. They also defined and discussed several neighborhood properties of γ-neighborhood, γ-neighborhood base at x, γ-closed neighborhood, γ-limit point, γ-isolated points. SarhadFaiqNamiq& Alias B. Khalaf[3],[4],[5],[6],[7],[8]. They study a new class of semi open sets, which they call a λ-open set and λ_c-open set in topological spaces and also they define the notions of λ-interior, λ-limit point, λ-derived set. In the second section, we define the notions of λ_c-interior, λ_c-limit point and λ_c-derived set of a set and they show that some of their properties are analogous to the properties in open sets. Moreover, we give some additional properties of λ_c-closure and λ_c-interior of a set.

3.Preliminaries

In 1963, Levine [9] defined semi open sets and semi continuous functions in a space *X*. SarhadFaiqNamiq and Alias B.Khalaf[3],[4],[5],[6], They introduce new classes of semi open sets called λ-open and λ_c-open sets in topological spaces. They consider λ as

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a function defined on the family of semi-open sets of X into the power set of X and $\lambda : SO(X) \to P(X)$ is called an s-operation if $V \subseteq \lambda(V)$ for each V.

The followings depended on[3],[4],[5],[6],[7],[8].

Definition 3.1

Let (X,τ) be a topological space and $\lambda : SO(X) \to P(X)$ be an s-operation. Then a subset A of X is called a λ-open set if for each $x \in A$ there exists a semi open set Usuch that $x \in U$ and $\lambda(U) \subseteq A$. The complement of a λ-open setis said to be λ-closed. The family of all λ-open (resp. λ-closed) subsets of a topological space (X,τ) is denotedby $SO_{\lambda}(X,\tau)$ or $SO_{\lambda}(X)$ (resp. $SC_{\lambda}(X,\tau)$ or $SC_{\lambda}(X)$).

Definition 3.2

A λ-open subset A of a topological space (X,τ) is called λ_c-open if for each $x \in A$ there exists a closed set F such that $x \in F \subseteq A$. The complement of a λ_c-open set is said to be λ_c-closed. The family of all λ_c-open (resp. λ_c-closed) subsets of a topological space (X,τ) is denotedby $SO_{\lambda c}(X,\tau)$ or $SO_{\lambda c}(X)$ (resp. $SC_{\lambda c}(X,\tau)$ or $SC_{\lambda c}(X)$).

Remark 3.3

From the definition of s-operation, it is clear that $\lambda(X) = X$ for any s-operation λ.Through out this thesis we assume that $\lambda(\phi) = \phi$, for any s-operation λ.

Definition 3.4

A subset of topological space (X,τ) is said to be λ_c-clopen if it is both λ_c-open and λ_c-closed set. The family of λ_c-clopen sets of X, denoted by $CO_{\lambda c}(X)$.

Definition 3.5

Let (X,τ) be a topological space and let Abe a subset of X. Then:

(1) The λ-closure of A ($\lambda Cl(A)$) is the intersection of all λ-closed sets containing A.

(2) The λ-interior of A ($\lambda Int(A)$) is the union of all λ_c-open sets of X contained in A.

(3) A point $x \in X$ is said to be a λ-limit point of A if every λ-open set containing x contains a point of A different from x, and the set of all λ-limit points of A is called the λ-derived set of A denoted by $\lambda d(A)$.

Proposition 3.6

Let (X,τ) be a topological space and $A \subseteq X$. For each point $x \in X$, $x \in \lambda Cl(A)$ if and only if $V \cap A \neq \phi$ for every $V \in SO_\lambda(X)$ such that $x \in V$.

Theorem 3.7

Suppose that $f:(X,\tau)\to(Y,\sigma)$ is (λ,γ)-continuous and (λ,γ)-closed function, then:

(1) For every g-λ_c-closed set A of (X,τ) the image $f(A)$ is a g-γ_c-closed set.

(2) For every g-γ_c-closed set B of (Y,σ) the inverse set $f^{-1}(B)$ is a g-λ_c-closed set.

Theorem 3.8

If $f:(X,\tau)\to(Y,\sigma)$ is a contra (λ,γ)-continuous function and satisfies the (λ,γ)-interiority condition, then f is (λ,γ)-continuous.

Theorem 3.9

Let $\lambda: SO(X) \to P(X)$ be an s-operation. Then for each $x \in X$, $\{x\}$ is λ_c-closed or $X \setminus \{x\}$ is g-λ_c-closed in (X,τ).

Theorem 3.10

If a subset A of a topological space (X,τ) is a g-λ_c-closed set in X,then $\lambda_c Cl(A) \setminus A$ does not contain any non empty λ_c-closed set in X.

Definition 3.11

A topological space (X,τ) is said to be:

(1) semi-T_0 [10] if for any distinct pair of points in X, there is a semi open set containing one of the points but not the other.

(2) semi-T_1 [10] if for any distinct pair of points x and y in X, there is a semi open set U in X containing x but not y and a semi open set V in X containing y but not x.

(3) semi-T_2 [10] if for any distinct pair of points x and y in X, there exist semi open sets U and V in X

containing x and y, respectively, such that $U \cap V = \phi$;

(4) semi-$T_{1/2}$- Space[11] if every sg-closed set is semi closed.

4.1 On λ_c-T_i Spaces ($i = 0, ½, 1, 2$)

Definition 4.1.1

Let (X, τ) be a topological space and $A \subseteq X$. Then the class of λ_c-open sets in A ($SO_{\lambda c}(A)$) is defined in a natural way as:

$SO_{\lambda c}(A) = \{A \cap V,\ V \in SO_{\lambda c}(X)\}$,

That is W is λ_c-open in A if and only if $W = A \cap V$, where V is a λ_c-open set in X.

Definition 4.1.2

A topological space (X, τ) is called a λ_c-T_0 space, if for each distinct points $x, y \in X$ there exists a λ_c-open set U contains one of them but not the other.

Example 4.1.3

Let $X=\{a,b,c\}$, and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as:

$$\lambda(A) = \begin{cases} A & \text{if } A = \{a\} \text{ or } \{b,c\} \text{ or } \{a,c\} \text{ or } \phi \\ X & \text{Otherwise} \end{cases}$$

.

Then a topological space (X, τ) is a λ_c-T_0 space.

Remark 4.1.4

Since every λ_c-open set is semi open, so every λ_c-T_0 space is a semi-T_0. But the converse is not true in general as it is seen in the following example:

Example 4.1.5

Let $X=\{a,b,c\}$, and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as:

$$\lambda(A) = \begin{cases} A & \text{if } A = \{a\} \text{ or } \phi \\ X & \text{Otherwise} \end{cases}.$$

Then a topological space (X, τ) is a semi-T_0 but it is not λ_c-T_0 space.

Theorem 4.1.6

A topological space (X, τ) is λ_c-T_0 if and only if for eachdistinct points x and y in X, $x \notin \lambda_c Cl(\{y\})$ or $y \notin \lambda_c Cl(\{x\})$.

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Proof. Let $x \neq y$ in a λ_c-T_0 space X. Then there exists a λ_c- open set U containing one of them but not the other, without loss of generality, we assume that U contains x but not y. Then $U \cap \{y\} = \phi$, this implies that $x \notin \lambda_c Cl(\{y\})$.

Conversely, let $x, y \in X$ such that $x \neq y$. Then by hypothesis $x \notin \lambda_c Cl(\{y\})$ or $y \notin \lambda_c Cl(\{x\})$. With out loss any of generality, we assume that $y \notin \lambda_c Cl(\{x\})$. Then $X \setminus \lambda_c Cl(\{x\})$ is an λ_c- open subset of X containing y. Since $x \in \lambda_c Cl(\{x\})$, then $x \notin X \setminus \lambda_c Cl(\{x\})$. So X is a λ_c-T_0 space.

Theorem 4.1.7

Every subspace of a λ_c-T_0 space X is a λ_c-T_0 space.

Proof. Let (X,τ) be a λ_c-T_0 space, and $A \subseteq X$. To show A is a λ_c-T_0 space. Let a, b be two distinct points of A. Since $A \subseteq X$, a, b are alsodistinct points of X. Since (X,τ) is a λ_c-T_0 space, there exists a λ_c- open set U in X such that $a \in U$and$b \notin U$ or $a \notin U$and $b \in U$. If $a \in U$and $b \notin U$,then $U \cap A$ is a λ_c- open set in A contain a notcontaining b, it follows that A is a λ_c-T_0 space. If $a \notin U$and $b \in U$ then similarly we get the result.

Theorem 4.1.8

The property of a space being a λ_c-T_0 space is preserved under a bijective and (λ,γ)-open functions.

Proof. Let (X,τ) be a λ_c-T_0 space and let f be a one to one (λ,γ)-open function of (X,τ) onto a topological space (Y,σ). Then we want to show that (Y,σ) is also γ_c-T_0. Let a, b be any two distinct points of Y. Since f is an onto function, there exist distinct points c, d of Xsuch that $f(c)=a$ and $f(d)=b$. Since (X,τ) is a λ_c-T_0 space, there exists a λ_c- open set U contain one of them, say c in X such that U does not contain d. Since f is a one to one (λ,γ)-open function, $f(U)$ is a γ_c- open set containing $f(c)=a$ and not containing $f(d)=b$. In other words, $f(U)$ is γ_c- open set containing a butnot b. Hence (Y,σ) is also γ_c-T_0.

Theorem 4.1.9

Let (X,τ) be a topological space and let (Y,σ) be a γ_c-T_0 space. Let $f:(X,\tau)\to(Y,\sigma)$ be a one to one, (λ,γ)-continuous function. Then (X,τ) is also a λ_c-T_0 space.

Proof. Let a,b be any two distinct points of X. Since f is one to one, and $a \neq b$, then $f(a) \neq (b)$. Let $c=f(a)$, $d=f(b)$ so that $a=f^{-1}(c)$ and $b=f^{-1}(d)$. Where c, $d \in Y$ such that $c \neq d$. Since$(Y,\sigma)$is a γ_c-T_0 space, there existsγ_c-open set H such that $c \in H$ but $d \notin H$. Since f is (λ,γ)-continuous, $f^{-1}(H)$ is λ_c-open. Now, since $c \in H$, then $f^{-1}(c) \in$

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$f^{-1}(H)$, so $a \in f^{-1}(H)$ but since f is one to one, so $b \notin f^{-1}(H)$. Hence (X,τ) is also a λ_c-T_0 space.

Corollary 4.1.10

If $f:(X,\tau)\to(Y,\sigma)$ is a one to one, contra (λ,γ)-continuous function and satisfies the (λ,γ)-interiority condition, and let (Y,σ) be a γ_c-T_0 space.

Then (X,τ) is also λ_c-T_0 space.

Proof. Follows from Theorem 3.8 and Theorem 4.1.9.

Definition 4.1.11

A topological space (X,τ) is said to be λ_c-$T_{1/2}$ space if every g-λ_c-closed set in (X,τ) is λ_c-closed.

Theorem 4.1.12

A topological space (X,τ) with an s-operation λ is λ_c-$T_{1/2}$ if and only if each singleton $\{x\}$ of X is λ_c-closed set or λ_c-open set.

Proof. Suppose $\{x\}$ is not λ_c-closed. Then by Proposition 3.9, $X\setminus\{x\}$ is g-λ_c-closed. Now since (X,τ) is λ_c-$T_{1/2}$, then $X\setminus\{x\}$ is λ_c-closed i.e. $\{x\}$ is λ_c-open.

Conversely, let A be any g-λ_c-closed set in (X,τ) and $x\in\lambda_c Cl(A)$. By hypothesis we have $\{x\}$ is λ_c-closed or λ_c-open. If $\{x\}$ is λ_c-closed. We have to show $x\in A$. For this, if we suppose that $x\notin A$, then $x\in\lambda_c Cl(A)\setminus A$ which is not possible by Proposition 3.10. Hence $x\in A$. Therefore, $\lambda_c Cl(A)=A$, i.e. A is λ_c-closed. So (X,τ) is λ_c-$T_{1/2}$. On the other hand, if $\{x\}$ is λ_c-open then as $x\in\lambda_c Cl(A)$, $\{x\}\cap A\neq\phi$. Hence $x\in A$. So A is λ_c-closed.

Theorem 4.1.13

Every λ_c-$T_{1/2}$ space is λ_c-T_0 space.

Proof. Let x and y be two distinct points of X. Then by Theorem 4.1.12, we have $\{x\}$ is λ_c-closed or λ_c-open. If $\{x\}$ is λ_c-closed set, then $\lambda_c Cl(\{x\})=\{x\}$, and since $y\neq x$, then $y\notin\lambda_c Cl(\{x\})$. Hence X is a λ_c-T_0 space, and if $\{x\}$ is λ_c-open, then the set $U=\{x\}$ is an λ_c-open subset of X which not contains y. Then $y\notin\lambda_c Cl(\{x\})$. Thus X is a λ_c-T_0 space by Theorem 4.1.6.

The converse of Theorem 4.1.13, is not true in general as shown by the following example.

Example 4.1.14

Let $X=\{a,b,c\}$, and $\tau = P(X)$.

We define an s-operation $\lambda : SO(X) \rightarrow P(X)$ as:

$$\lambda(A)=\begin{cases} A & \text{if } A=\{a\} \text{ or } \{a,b\} \text{ or } \phi \\ X & \text{Otherwise} \end{cases}$$

.

Then a topological space (X,τ) is a λ_c-T_0 space but not a λ_c-$T_{1/2}$ space because $\{a,c\}$ is g-λ_c-closed but not λ_c-closed

Remark 4.1.15

It is clear from the definitions that, every λ_c-$T_{1/2}$ is semi-$T_{1/2}$, but the converse is not true in general, we refer to Example 4.1.14, in which (X,τ) is a semi-$T_{1/2}$, but not λ_c-$T_{1/2}$ space.

Theorem 4.1.16

If a topological space (X,τ) is λ_c-$T_{1/2}$, then every subset of X is theintersection of all λ_c-open sets and all λ_c-closed sets that containing it.

Proof. Let (X,τ) be a λ_c-$T_{1/2}$ space with $F \subseteq X$ arbitrary. Then F = $\bigcap \{ X \setminus \{x\} : x \notin F \}$ is an intersection of λ_c-open sets and λ_c-closed sets by Theorem 4.1.12. The result follows.

Theorem 4.1.17

Every subspace of a λ_c-$T_{1/2}$ space X is a λ_c-$T_{1/2}$ space.

Proof. Let (X,τ) be a λ_c-$T_{1/2}$ space, and $A \subseteq X$. To show A is a λ_c-$T_{1/2}$ space. Let $y \in A$. Since $A \subseteq X$, then $y \in X$ and by Theorem 4.1.12, $\{y\}$ is a λ_c-closed set or λ_c-open set. Implies that $\{y\} \cap A = \{y\}$ is a λ_c-closed set or λ_c-open set in A. Then by Theorem 4.1.12, A is a λ_c-$T_{1/2}$ space.

Theorem 4.1.18

Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a (λ,γ)-continuous and (λ,γ)-closed function. Then:

(1) If f is injective and (Y,σ) is a γ_c-$T_{1/2}$ space, then (X,τ) is a λ_c-$T_{1/2}$ space.

(2) If f is surjective and (X,τ) is a λ_c-$T_{1/2}$ space, then (Y,σ) is a γ_c-$T_{1/2}$ space.

Proof.(**1**) Let A be a g-λ_c-closed set in (X,τ). To show that A is λ_c-closed. By Theorem 3.7,we have $f(A)$ is g-γ_c-closed. Since (Y,σ) is γ_c-$T_{1/2}$, $f(A)$ is a γ_c-closed set. Since f is injective and (λ,γ)-continuous, $f^{-1}(f(A)) = A$ is a λ_c-closed set in X. Hence (X,τ) is a λ_c-$T_{1/2}$ space.

(**2**) Let B be a g-γ_c-closed set in (Y,σ). By Theorem 3.7, $f^{-1}(B)$ is g-λ_c-closed. Since (X,τ) is a λ_c-$T_{1/2}$ space, $f^{-1}(B)$ is λ_c-closed. Since f is surjective and (λ,γ)-continuous, $f(f^{-1}(B)) = B$ is a γ_c-closed set in Y. Therefore (Y,σ) is γ_c-$T_{1/2}$.

Definition 4.1.19

A topological space (X,τ) is called a λ_c-T_1 space, if for each distinct points $x, y \in X$, there exist λ_c-open sets U,V containing x and y respectively such that $y \notin U$ and $x \notin V$.

Theorem 4.1.20

A topological space (X,τ) is a λ_c-T_1 space if and only if every singleton subset $\{x\}$ of X is λ_c-closed.

Proof. Let x and y be two distinct points of X. Then $X\setminus\{x\}$ is a λ_c-open set which contains y but does not contain x. Similarly $X\setminus\{y\}$ is a λ_c-open set which contains x but does not contain y. Hence the space (X,τ) is λ_c-T_1.

Conversely, let x be any point of X. We want to show $\{x\}$ is λ_c-closed, that is, to show that $X\setminus\{x\}$ is λ_c-open set. Let $y \in X\setminus\{x\}$. Then $y \neq x$. Since X is λ_c-T_1, there exists a λ_c-open set H such that $y \in H$ but $x \notin H$. It follows that $y \in H \subseteq X\setminus\{x\}$. Hence $\{x\}$ is λ_c-closed.

Remark 4.1.21

From the definition of λ_c-T_1 and semi-T_1 space, every λ_c-T_1 space is semi-T_1, but the converse is not true in general, clearly in Example 4.1.14, (X,τ) is a semi-T_1, but not λ_c-T_1 space.

Proposition 4.1.22

If $\lambda : SO(X) \to P(X)$ is a λ-regular s-operation. Then a topological space (X,τ) is a λ_c-T_1 space, if and only if every finite subset of X is λ_c-closed.

Proof. Obvious.

Theorem 4.1.23

Every λ_c-T_1 space is a λ_c-$T_{1/2}$ space.

Proof. Suppose that $A$is a set which is not λ_c-closed, then there exists $x \in \lambda_c Cl(A) \setminus A$, so $\{x\} \subseteq \lambda_c Cl(A) \setminus A$ and $\{x\}$ is λ_c-closed, since we are in a λ_c-T_1space, and as $\{x\} \neq \phi$, by Theorem 3.10, $A$is not g-λ_c-closed.

The converse of Theorem 4.1.23, is not true in general and we can show it by the following example.

Example 4.1.24

Let $X = \{a,b\}$, and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as:

$$\lambda(A) = \begin{cases} A & \text{if } \ A = \{a\} \text{ or } \phi \\ X & \text{Otherwise} \end{cases}.$$

Then a topological space (X,τ) is a λ_c-$T_{1/2}$ space but not λ_c-T_1 space.

Theorem 4.1.25

The property of a space being a λ_c-T_1 space is preserved under bijective and (λ,γ)-open functions.

Proof. Let (X,τ) be a λ_c-T_1 space and let f be a one to one (λ,γ)-open function of (X,τ) onto a topological space (Y,σ). Then we want to show that (Y,σ) is also a γ_c-T_1. Let a, b be any two distinct points of Y. Since f is an onto function, there distinct points c, d of Xexistsuch that $f(c)=a$ and $f(d)=b$. Since (X,τ) is a λ_c-T_1 space, there exist λ_c-open sets U and V such that $c \in U$ but $d \notin U$ and $d \in V$ but $c \notin V$. Since f is a one to one (λ,γ)-open function, $f(U)$ and $f(V)$ are γ_c-open sets containing $a = f(c) \in f(U)$ but $b = f(d) \notin f(U)$ and $b = f(d) \in f(V)$ but $a = f(c) \notin f(V)$. Hence (Y,σ) is alsoa γ_c-T_1 space.

Theorem 4.1.26

Let (X,τ) be a topological space and let (Y,σ) be a γ_c-T_1 space. If $f:(X,\tau) \to (Y,\sigma)$ is a one to one and (λ,γ)-continuous function.Then (X,τ) is also a λ_c-T_1 space.

Proof. Let a,b be any two distinct points of X. Since f is one to one and $a \neq b$, then $f(a) \neq f(b)$. Let $c = f(a)$, $d = f(b)$ so that $a = f^{-1}(c)$ and $b = f^{-1}(d)$. Where $c, d \in Y$ such that $c \neq d$. Since (Y,σ) is γ_c-T_1space, there exist γ_c-open sets H and K such that $c \in H, d \notin H, d \in K$ and $c \notin K$. Since f is (λ,γ)-continuous, $f^{-1}(H)$ and $f^{-1}(K)$ are λ_c-open sets. Now $c \in H, d \notin H, d \in K$ and $c \notin K$. Then by hypothesis $f^{-1}(c) \in f^{-1}(H), f^{-1}(d) \notin f^{-1}(H), f^{-1}(d) \in f^{-1}(K)$ and $f^{-1}(c) \notin f^{-1}(K)$. So $a \in f^{-1}(H), b \notin f^{-1}(H), b \in f^{-1}(K)$ and $a \notin f^{-1}(K)$. Hence (X,τ) is also λ_c-T_1 space.

Corollary 4.1.27

Let (X,τ) be a topological space and let (Y,σ) be a γ_c-T_1 space. If $f:(X,\tau)\to(Y,\sigma)$ is a one to one and (λ,γ)-continuous function. Then (X,τ) is a λ_c-T_i space, for $i=$ 0, ½.

Proof. Follows from Theorem 4.1.26, and also we have every λ_c-T_1 space is λ_c-T_i space, for $i=$ 0, ½.

Corollary4.1.28

If $f:(X,\tau)\to(Y,\sigma)$ is a one to one, contra (λ,γ)-continuous function and satisfies the (λ,γ)-interiority condition, and let (Y,σ) be a γ_c-T_1 space. Then (X,τ) is a λ_c-T_i space, for $i=$ 0, ½.

Proof. Follows from Theorem 3.8 and Corollary 4.1.27.

Theorem 4.1.29

Let $f:(X,\tau)\to(Y,\sigma)$ be an onto (λ,γ)-closed function. If X is a λ_c-T_1 space, then so is Y.

Proof. Let $y \in Y$. Then there exists $x \in X$ such that $f(x)=y$. Since $x \in X$ and X is a λ_c-T_1 space, then by Theorem 4.1.20, $\{x\}$ is λ_c-closed. Since f is (λ,γ)-closed, $f(\{x\})=\{y\}$ γ_c-closed. Hence by Theorem 4.1.20, Y is a γ_c-T_1 space.

Theorem 4.1.30

Every subspace of a λ_c-T_1 space X is a λ_c-T_1 space.

Proof. Let (X,τ) be a λ_c-T_1 space, and $A \subseteq X$. Let $a \in A$. Then $a \in X$ and $\{a\}$ is λ_c-closed in X (Since (X,τ) is λ_c-T_1 space). Therefore $\{a\}=\{a\}\cap A$ is λ_c- closed A. Thus by Theorem 4.1.20, $A$is a λ_c-T_1 space.

Definition 4.1.31

A topological space (X,τ) is called a λ_c-symmetric space, if for x and y in X, $x \in \lambda_c Cl(\{y\})$ implies that $y \in \lambda_c Cl(\{x\})$.

Theorem 4.1.32

Let (X,τ) be a λ_c-symmetric space. Then the following statements are equivalent:

(1) (X,τ) is λ_c-T_0.

(2) (X,τ) is λ_c-$T_{1/2}$

(3) (X,τ) is λ_c-T_1.

Proof. It is enough to prove only that (1) gives (3). Let $x \neq y$ and since (X,τ) is λ_c-T_0, we may assume that $x \in U \subseteq X \setminus \{y\}$ for some $U \in SO_{\lambda c}(X)$. Then $x \notin \lambda_c Cl(\{y\})$ and hence $y \notin \lambda_c Cl(\{x\})$. Therefore, there exists $V \in SO_{\lambda c}(X)$ such that $y \in V \subseteq X \setminus \{x\}$ and (X,τ) is a λ_c-T_1 space.

4.2. Some properties of separation axioms via λ_c- open set

Theorem 4.2.1

Let (X,τ) be a λ_c-symmetric topological space and let (Y,σ) be a γ_c-T_0 space. If $f:(X,\tau)\to(Y,\sigma)$ is a one to one and (λ,γ)-continuous function. Then (X,τ) is λ_c-T_i space, for $i=$ 0, ½, 1.

Proof. Follows from Theorem 4.1.9 and Theorem 4.1.32.

Theorem 4.2.2

Let (X,τ) be a λ_c-symmetric topological space and let (Y,σ) be a γ_c-$T_{1/2}$ space. If $f:(X,\tau)\to(Y,\sigma)$ be a one to one ,(λ,γ)-continuous and (λ,γ)-closed function. Then (X,τ) is λ_c-T_i space, for $i=$ 0, ½, 1.

Proof. Follows from Theorem 4.1.18 andTheorem 4.1.32.

Definition 4.2.3

A topological space (X,τ) is called a λ_c-T_2 space if for each two distinct points $x, y \in X$ there exist λ_c-open sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Remark 4.2.4

From the definition of λ_c-T_2 and semi-T_2 space, every λ_c-T_2 is semi-T_2, but the converse is not true in general, clearly in Example 4.1.14, (X,τ) is a semi-T_2, but not λ_c-T_2 space.

Theorem 4.2.5

Every λ_c-T_2 space is λ_c-T_1 space.

Proof. Obvious.

The converse of Theorem 4.2.5, is not true in general and we can show it by the following example.

Example 4.2.6

Let $X=\{a,b,c\}$, and $\tau=P(X)$.

We define an s-operation $\lambda: SO(X)\to P(X)$ as:

$$\lambda(A)=\begin{cases} A & \text{if } A=\{a,b\} \text{ or } \{a,c\} \text{ or } \{b,c\} \text{ or } \phi \\ X & \text{Otherwise}\end{cases}$$

.

Clearly (X,τ) is λ_c-T_1 space, but it is not λ_c-T_2.

Theorem 4.2.7

The property of a space being a λ_c-T_2 space is preserved under bijective and (λ,γ)-open functions.

Proof. Let (X,τ) be a λ_c-T_2 space and let f be a one to one (λ,γ)-open function of (X,τ) onto another topological space (Y,σ). Then we want to show that (Y,σ) is also γ_c-T_2. Let *a*, *b* be any two distinct points of *Y*. Since f is an onto function, there distinct points *c*, *d* of *X*existsuch that $f(c)=a$ and $f(d)=b$. Since (X,τ) is a λ_c-T_2 space, there exist disjoint λ_c-open sets *U* and *V* such that $c\in U$ but $d\notin U$ and $d\in V$ but $c\notin V$. Since f is a one to one (λ,γ)-open function, $f(U)$ and $f(V)$ are γ_c-open sets containing $a = f(c)\in f(U)$ but $b = f(d)\notin f(U)$ and $b = f(d)\in f(V)$ but $a = f(c)\notin f(V)$. And also we have $U\cap V=\phi$ and since *f* is one to one, this implies that $f(U)\cap f(V)=\phi$. Hence (Y,σ) is alsoa γ_c-T_2 space.

Theorem 4.2.8

Let (X,τ) be a topological space and let (Y,σ) be a γ_c-T_2 space. If $f:(X,\tau)\to(Y,\sigma)$ is a one to one (λ,γ)-continuous function.Then (X,τ) is λ_c-T_2.

Proof. Let a,b be any two distinct points of X. Since *f* is one to one and $a\neq b$ then $f(a)\neq f(b)$. Let $c=f(a)$, $d=f(b)$ are distinct so that $a=f^{-1}(c)$ and $b=f^{-1}(d)$. Where c, $d\in Y$. Since (Y,σ) is a γ_c-T_2 space, there exist γ_c-open sets *H* and *K* such that $c\in H, d\in K$ and $H\cap K=\phi$. Since f is (λ,γ)-continuous, $f^{-1}(H)$

and $f^{-1}(K)$ are λ_c-open. Now $f^{-1}(H) \cap f^{-1}(K)$ = $f^{-1}(H \cap K) = f^{-1}(\phi) = \phi,$ and $c \in H$ then $f^{-1}(c) \in f^{-1}(H),$ so $a \in f^{-1}(H)$ and $d \in K,$ then $f^{-1}(d) \in f^{-1}(K),$ so $b \in f^{-1}(K).$ Hence (X,τ) is also a λ_c-T_2 space.

Corollary 4.2.9

Let (X,τ) be a topological space and let (Y,σ) be a γ_c-T_2 space. If $f:(X,\tau) \to (Y,\sigma)$ is a one to one and (λ,γ)-continuous function. Then (X,τ) is a λ_c-T_i space, for $i =$ 0, ½, 1.

Proof. Follows from Theorem 4.2.8, and also we have every λ_c-T_2 space is λ_c-T_i space, for $i =$ 0, ½, 1.

Corollary 4.2.10

Let (X,τ) be a topological space and let (Y,σ) be a γ_c-T_2 space. If $f:(X,\tau) \to (Y,\sigma)$ is a one to one and (λ,γ)-continuous function. Then (X,τ) is semi-T_i space, for $i =$ 0, ½, 1,2.

Proof. Follows from Theorem 4.2.8 andCorollary 4.2.9, and also we have every λ_c-T_2 space is semi-T_i space, for $i =$ 0, ½, 1,2.

Corollary 4.2.10

Let (X,τ) be a topological spaceand let (Y,σ) be a γ_c-T_2space. If f : $(X,\tau) \to (Y,\sigma)$ is a one to one and contra (λ,γ)-continuous function and satisfies the (λ,γ)-interiority condition.Then (X,τ) is also λ_c-T_2 space.

Proof. Follows from Theorem 3.8 and Theorem 4.2.8.

Theorem 4.2.11

Every subspace of a λ_c-T_2 space *X* is a λ_c-T_2 space.

Proof. Let (X,τ) be a λ_c-T_2 space, and *A*⊆*X*. To show *A* is a λ_c-T_2 space. Let *a*, *b* be two distinct points of *A*. Since *A* ⊆*X*, so*a*, *b* are alsodistinct points of *X*. Since (X,τ) is a λ_c-T_2 space, there exist disjoint λ_c-open sets *U* and *V* in *X* contain*a* and *b* respectively. $G = U \cap A$ and $H = V \cap A$ are λ_c- open sets in *A* contain *a* and *b* respectively, $G \cap H = (U \cap A) \cap (V \cap A) = (U \cap V) \cap A = \phi \cap A = \phi$. Hence *A* is a λ_c-T_2 space.

Theorem 4.2.12

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If X is a λ_c-T_2 space, then for any two distinct points $a, b \in X$, there are λ_c-closed sets A and B such that $a \in A$, $b \notin A$, $a \notin B$, $b \in B$ and $X = A \cup B$.

Proof. Since X is λ_c-T_2 space, then for any distinct $a, b \in X$, there exist λ_c-open sets U and V such that $a \in U$, $b \in V$ and $U \cap V = \phi$. Therefore $U \subseteq X \setminus V$ and $V \subseteq X \setminus U$. Hence $a \in X \setminus V$. Put $X \setminus V = A$. This gives $a \in A$ and $b \notin A$. Also $b \in X \setminus U$. Put $X \setminus U = B$. Therefore, $b \in B$ and $a \notin B$. Moreover $A \cup B = (X \setminus U) \cup (X \setminus V) = X$.

Theorem 4.2.13

If X is a λ_c-T_2 space, then for every point a of X, $\{a\} = \bigcap C_a$, where C_a is a λ_c-closed set containing a λ_c-open set U which contains a.

Proof. Since X is a λ_c-T_2 space, therefore for any a, b such that $a \neq b$, there exist λ_c-open sets U and V such that $a \in U$, $b \in V$ and $U \cap V = \phi$. This gives $U \subseteq X \setminus V$. Since $X \setminus V$ is λ_c-closed and $U \subseteq X \setminus V = C_a$, a λ_c-closed set which contains a and does not contain b. Since b is an arbitrary point of X different from a, then $b \notin \bigcap C_a$. Thus a is the only point which is in every λ_c-closed which contains a, that is, $\{a\} = \bigcap C_a$. Hence the proof.

Definition 4.2.14

Let (X, τ) be a topological space, A sequence $\{x_k\}$ is said to λ_c-converge to a point $x \in X$, denoted $x_k \xrightarrow{\lambda c} x$, if for every λ_c-open set U containing x, there exists a positive integer n such that $x_k \in U$ for all $k \geq n$.

Now we prove the following:

Theorem 4.2.15

Let X be a λ_c-T_2 space. Then any sequence in X can λ_c-converge to at most one point.

Proof. Let $\{x_k\}$ be a sequence in X which is λ_c-converging to x and y. Then by definition of λ_c-T_2 space, there exist λ_c-open sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Since $x_k \xrightarrow{\lambda c} x$ therefore there exists a positive integer n_1 such that $x_k \in U$, for all $k \geq n_1$. Also $x_k \xrightarrow{\lambda c} y$, therefore there exists a positive integer n_2 such that $x_k \in V$, for all $k \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then $x_k \in U$, and $x_k \in V$, for all $k \geq n_0$ or $x_k \in U \cap V$, for all $k \geq n_0$. This contradiction proves that $\{x_k\}$ λ_c-converges to at most one point.

Example 4.2.16

Let $X = \{a, b, c\}$ and $\tau = P(X)$. We define s-operation $\lambda : SO(X) \to P(X)$ by $\lambda(A) = X$ for $\phi \neq A \subseteq X$. Then $SO_{\lambda c}(X) = \{\phi, X\}$, then the sequence $\{x_k\}$ where $x_k = x$ for each k, is λ_c-converging to *y*, for all $y \in X$.

Remark 4.2.17

From Definitions 3.11, 4.1.2, 4.1.11, 4.1.19 and 4.2.3, Examples 4.1.5, 4.1.14, 4.1.27 and 4.2.6, and Theorems 4.1.13, 4.1.23 and 4.2.35, we get the following implications:

λ_c-$T_2 \longrightarrow \lambda_c$-$T_1 \longrightarrow \lambda_c$-$T_{1/2} \longrightarrow \lambda_c$-$T_0$

$\downarrow \downarrow \downarrow \downarrow$

semi-$T_2 \longrightarrow$ semi -$T_1 \longrightarrow$ semi -$T_{1/2} \longrightarrow$ semi -T_0.

Proposition 4.2.18

Let $f : (X, \tau) \to (Y, \sigma)$ be a (λ, γ)-homeomorphism. If (X, τ) is λ_c-$T_{1/2}$ then (Y, σ) is γ_c-$T_{1/2}$.

Proof. It is follows from Theorem 4.1.18.

Corollary 4.2.19

If $f : (X, \tau) \to (Y, \sigma)$ is a one to one, contra (λ, γ)-continuous function and satisfies the (λ, γ)-interiority condition, and let (Y, σ) be a γ_c-T_2 space. Then (X, τ) is a λ_c-T_i space, for $i =$ 0, ½, 1.

Proof. Follows from Theorem 3.8 and Corollary 4.2.9.

Corollary 4.2.20

If $f : (X, \tau) \to (Y, \sigma)$ is a one to one, contra (λ, γ)-continuous function and satisfies the (λ, γ)-interiority condition, and let (Y, σ) be a γ_c-T_1 space.

Then (X, τ) is also λ_c-T_1 space.

Proof. Follows from Theorem 3.8 and Theorem 4.1.2

References

1. Ahmad B., H.S., *Properties of Gamma-operations in topological spaces.* Aligarh Bull. Math, 2003. **22**(1): p. 45-51.
2. Kasahara, S., *Operation-compact spaces.* Math. Japonica, 1979. **24**(1): p. 97-105.
3. Namiq, S.F., *NEW TYPES OF CONTINUITY AND SEPARATION AXIOMS BASED ON OPERATIONS IN TOPOLOGICAL SPACES*. 2011, University of Sulaimani.
4. Khalaf, A.B. and S.F. Namiq, *[[lambda]. sub. c]-open sets and [[lambda]. sub. c]-separation axioms in topological spaces.* Journal of Advanced Studies in Topology, 2013. **4**(1): p. 150-159.
5. Namiq, S.F., *lambda–Connected Spaces Via lambda-Open Sets.* Garmian University, 2015. **1**: p. 165-178.
6. Namiq, S.F., *ON Minimal λ*-Open Sets.* International Journal of Scientific & Engineering Research. **5**(10).
7. Namiq, A.B.K.a.S.F., *ON Minimal λc-Open Sets Via λc-Open Sets.* Asian Journal of Fuzzy and Applied Mathematics 2014. **02**(06).
8. Khalaf, A.B. and S.F. Namiq, *Generalized λ-Closed Sets and-Continuous Functions.* International Journal of Scientific & Engineering Research, 2012. **03**(12).
9. Levine, N., *Semi-open sets and semi-continuity in topological spaces.* The American Mathematical Monthly, 1963. **70**(1): p. 36-41.
10. Maheshwari, S. and R. Prasad, *Some new separation axioms.* ANNALES DE LA SOCIETE SCIENTIFIQUE DE BRUXELLES SERIES 1-SCIENCES MATHEMATIQUES ASTRONOMIQUES ET PHYSIQUES, 1975. **89**(3): p. 395-402.
11. Bhattacharyya, P. and B. Lahiri, *Semi-generalized closed sets in topology*. Indian J. Math, 1987. **29**(3): p. 375-382.